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Exact Order of Subsets of Asymptotic Bases in Additive Number Theory

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Let A be an asymptotic basis of order h. Define

$$I_k(A) = \{F \mid F \subseteq A, |F| = k \text{ and } A \setminus F \text{ is a basis}\},$$

where $|F|$ indicates the number of elements in the finite set F. We denote by $g(A)$ the exact order of A. Define

$$G_k(h) = \max_{\substack{A \\ g(A) \leqslant h}} \max_{F \in I_k(A)} g(A \setminus F).$$

An open problem in additive number theory is to calculate $G_k(h)$. We prove in this paper that

$$G_k(h) \geqslant (4/3)(h/(k+1))^{k+1} + O(h^k)$$

as h tends to infinity, which is an improvement of a result of Melvyn B. Nathanson (The exact order of subsets of additive bases, *in* "Proceedings, New York Number Theory Seminar, 1982," Lecture Notes in Mathematics, Vol. 1052, pp. 273–277, Springer-Verlag, 1984). On the other hand, we estimate $G_k(h)$ as k tends to infinity for any fixed integer h. It is proved that $G_k(h)$ has order of magnitude k^{h-1} as k tends to infinity for any fixed $h \geqslant 2$. © 1988 Academic Press, Inc.

1. INTRODUCTION

Let A be a set of integers, and let hA denote the set of all sums of h not necessarily distinct elements of A. If hA contains all sufficiently large integers, then A is an *asymptotic basis* of order h. The set A is an asymptotic basis if A is an asymptotic basis of order h for some h. If A is an asymptotic basis, the *exact order* of A, denoted $g(A)$, is the smallest integer h such that A is an asymptotic basis of order h.

Let A be an asymptotic basis, $a \in A$. It is not necessary that $A \setminus \{a\}$ is also an asymptotic basis. We denote by I the set of elements $a \in A$ such that

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$A \setminus \{a\}$ is an asymptotic basis. For any $a \in I$, Erdös and Graham [1] investigated how large $g(A \setminus \{a\})$ can be in terms of $g(A)$. Let

$$G_1(h) = \max_{\substack{A \\ g(A \leqslant h)}} \lim_{a \in I} g(A \setminus \{a\}).$$

Erdös and Graham [1] proved in 1980 that

$$\tfrac{1}{4}(1 + o(1))\, h^2 \leqslant G_1(h) \leqslant (5/4)(1 + o(1))\, h^2.$$

In his doctoral thesis, G. Grekos [2] improved this estimate to

$$\tfrac{1}{3}h^2 + O(h) \leqslant G_1(h) \leqslant h^2 + h,$$

where the upper bound holds for any $h > 1$. Recently, Nash obtained in his doctoral thesis [6] that

$$G_1(h) \leqslant \tfrac{1}{2}h^2 + O(h)\ .$$

However, it is an open problem to improve the estimates of $G_1(h)$ (see [7] or [6]).

In 1982, Nathanson considered the general form of this problem. Instead of cancelling only one element in A, he cancelled a finite subset of A with fixed cardinality.

Let $k \geqslant 1$ be an integer. If A is an asymptotic basis, let $I_k(A)$ denote the set of all subsets $F \subseteq A$ such that F has cardinality k and the set $A \setminus F$ is an asymptotic basis. Define

$$G_k(h) = \max_{\substack{A \\ g(A) \leqslant h}} \max_{F \in I_k(A)} g(A \setminus F).$$

Nathanson [7] proved that

$$G_k(h) \geqslant ([h/(k+1)] + 1)^{k+1} - 1,$$

where $h > k$ and $[x]$ indicates the largest integer not exceeding x. In Section 2, we improve this estimate to

$$G_k(h) \geqslant \tfrac{4}{3}(h/(k+1))^{k+1} + O(h^k).$$

In case $k = 1$, this is the lower bound of Grekos.

Professor Nathanson [7] thinks that it is significant to estimate $G_k(h)$ as k tends to infinity for any fixed integer $h \geqslant 2$. Nash [5] proved that $G_k(2) = 2k + 2$ for any $k \geqslant 1$. In Section 3, we prove that $G_k(h)$ has order of magnitude k^{h-1} as k tends to infinity for any fixed $h \geqslant 2$, namely

$$G_k(h)+1 \geqslant \frac{2}{(h-1)^{h-1}}k^{h-1}+\frac{4h-5}{(h-1)^{h-2}}k^{h-2}+O(k^{h-3}),$$

$$G_k(h)+1 \leqslant \frac{2}{(h-1)!}k^{h-1}+\frac{h-1}{(h-2)!}k^{h-2}+O(h^{h-3}).$$

If $h=2$ this is the result of Nash.

2. Estimate of the Lower Bound of $G_k(h)$ for Any k

Lemma 2.1. *Let $h \geqslant k+1$ and $k \geqslant 1$, and*

$$\begin{aligned} &u=[h/(k+1)], \qquad u'=[2h/(3k+3)],\\ &\sigma=h-(k-1)\,u-2u',\\ &b_1=[2h/(k+1)],\\ &b_i=ub_{i-1}+\sigma, \qquad \textit{for} \quad i=1,2,...,k,\\ &d=u'b_k+b_1-h+(k-1)\,u+u'. \end{aligned}$$

Then

(i) $md+\sum_{i=1}^k x_ib_i+h-\sum_{i=1}^k x_i \geqslant (m-1)\,d+\sum_{i=1}^k x_ib_i+b_1+u'b_k$;

(ii) $(m-1)\,d+\sum_{i=1}^k x_ib_i+u'b_k+h-\sum_{i=1}^k x_i-u' \geqslant md+\sum_{i=1}^k x_ib_i$

hold for any $m \geqslant 1$, and

$$0 \leqslant x_i \leqslant u \qquad \textit{for} \quad i=1,2,...,k-1, \qquad 0 \leqslant x_k \leqslant u'.$$

Proof. From the definitions, we have

$$\begin{aligned} md&+\sum_{i=1}^k x_ib_i+h-\sum_{i=1}^k x_i\\ &=(m-1)\,d+\sum_{i=1}^k x_ib_i+h-\sum_{i=1}^k x_i+d\\ &=(m-1)\,d+\sum_{i=1}^k x_ib_i+b_1+u'b_k+\sum_{i=1}^{k-1}(u-x_i)+(u'-x_k)\\ &\geqslant (m-1)\,d+\sum_{i=1}^k x_ib_i+b_1+u'b_k, \end{aligned}$$

which shows (i). It follows from the definitions that

$$\begin{aligned}(m-1)\,d+\sum_{i=1}^{k} x_i b_i+u'b_k+h-\sum_{i=1}^{k} x_i-u' &= md-u'b_k-b_1+h-(k-1)\,u-u'+\sum_{i=1}^{k} x_i b_i\\ &\quad+u'b_k+h-\sum_{i=1}^{k} x_i-u'\\ &= md+\sum_{i=1}^{k} x_i b_i+2h-(k-1)\,u-2u'-\sum_{i=1}^{k} x_i-b_1\\ &\geqslant md+\sum_{i=1}^{k} x_i b_i+2h-2(k-1)\,u-3u'-b_1.\end{aligned}$$

Observing that

$$\begin{aligned}2(k-1)\,u+3u'+b_1 \leqslant 2(k-1)\frac{h}{k+1}+3\,\frac{2h}{3k+3}+\frac{2h}{k+1} &= 2h\left(\frac{k-1}{k+1}+\frac{1}{k+1}+\frac{1}{k+1}\right)\\ &= 2h,\end{aligned}$$

we have

$$(m-1)\,d+\sum_{i=1}^{k} x_i b_i+2h-2(k-1)\,u-3u'-b_1 \geqslant md+\sum_{i=1}^{k} x_i b_i,$$

which implies (ii). The proof is complete.

LEMMA 2.2. *Let $h, k, u, u', b_1, ..., b_k$ be as in Lemma 1.1. If*

$$0 \leqslant x_i \leqslant u \quad \text{for} \quad i=1, 2, ..., k-1, \quad 0 \leqslant x_k \leqslant 2u',$$

then

$$\sum_{i=\lambda+1}^{k} x_i b_i+ub_\lambda+h-\sum_{i=\lambda+1}^{k} x_i-u \geqslant \sum_{i=\lambda+1}^{k} x_i b_i+b_{\lambda+1}$$

holds for any $1 \leqslant \lambda \leqslant k-1$.

Proof. Noting that $\lambda \geqslant 1$, we have

$$\begin{aligned}h-\sum_{i=\lambda+1}^{k} x_i-u &\geqslant h-(k-2)\,u-2u'-u\\ &= h-(k-1)\,u-2u'\\ &= \sigma.\end{aligned}$$

Therefore

$$\sum_{i=\lambda+1}^{k} x_i b_i + ub_\lambda + h - \sum_{i=\lambda+1}^{k} x_i - u \geqslant \sum_{i=\lambda+1}^{k} x_i b_i + ub_\lambda + \sigma$$
$$= \sum_{i=\lambda+1}^{k} x_i b_i + b_{\lambda+1},$$

which proves the lemma.

LEMMA 2.3. *Let d be as in Lemma* 1.1. *Then*

$$D = \{a \mid a \equiv 0 \text{ or } 1 \pmod{d}\}$$

is an asymptotic basis with

$$g(D) = \tfrac{4}{3}(h/(k+1))^{k+1} + O(h^k).$$

Proof. $h \geqslant 3k+3$ implies $d > 2$. Let n be any positive integer with

$$n = qd + r,$$

where $0 \leqslant r < d$. If $r = 0$ then $n = qd \in D \subseteq (d-1)D$. If $r > 0$ then

$$n = (qd+1) + (r-1) \in rD \subseteq (d-1)D.$$

Hence D is an asymptotic basis with the exact order $g(D) < d$.

For any s, if

$$sd + (d-1) = \sum_{i=1}^{t} a_i, \qquad a_i \in D,$$

then it follows from the definition of D that there exist at least $d-1$ elements a_i of $\{a_1, a_2, ..., a_t\}$ such that $a_i = 1 \pmod{d}$, which implies that $t \geqslant d-1$. Therefore

$$\begin{aligned}
g(D) &= d-1 \\
&= u'b_k + b_1 - h + (k-1)u + u' - 1 \\
&= u'ub_{k-1} + u'\sigma + b_1 - h + (k-1)u + u' - 1 \\
&= \cdots \\
&= u'u^{k-1}b_1 + u'(u^{k-2} + \cdots + u + 1) + b_1 - h + (k-1)u + u' - 1 \\
&= \frac{2h}{3k+3}\left(\frac{h}{k+1}\right)^{k-1} \frac{2h}{k+1} + O(h^k) \\
&= \frac{4}{3}\left(\frac{h}{k+1}\right)^{k+1} + O(h^k).
\end{aligned}$$

The proof is complete.

THEOREM 2.4. *Let $k \geqslant 1$. Then*

$$G_k(h) \geqslant \frac{4}{3}\left(\frac{h}{k+1}\right)^{k+1} + O(h^k)$$

as h tends to infinity.

Proof. Taking

$$F = \{b_1, b_2, ..., b_k\}, \qquad A = F \cup D,$$

where $b_1, b_2, ..., b_k$, d, and D are as in Lemmas 1.1 and 1.2, we obtain with Lemma 1.3 that A is an asymptotic basis. Now we assume $h \geqslant 3k+3$. Then

$$4 \leqslant b_1 < b_2 < \cdots < b_k < d,$$

which shows $A \backslash F = D$, so that $F \in I_k(A)$. By Lemma 1.3, it is sufficient to prove $g(A) \leqslant h$.

Let $m \geqslant$ and

$$0 \leqslant x_i < u \qquad \text{for} \quad i = 1, 2, ..., k-1, \quad 0 \leqslant x_k \leqslant u'.$$

Since

$$\begin{aligned} 2 + \sum_{i=1}^{k} x_i + u' &\leqslant 2 + (k-1)\,u + 2u' \\ &\leqslant 2 + (k-1)\,h/(k+1) + 4h/(3k+3) \\ &= h - (2h/(3k+3) - 2) \\ &\leqslant h, \end{aligned}$$

we have

$$md + \sum_{i=1}^{k} x_i b_i \in hA,$$

$$(m-1)\,d + \sum_{i=1}^{k} x_i b_i + b_1 + u'b_k \in hA.$$

Thus, any x satisfying

$$md + \sum_{i=1}^{k} x_i b_i \leqslant x \leqslant md + \sum_{i=1}^{k} x_i b_i + h - \sum_{i=1}^{k} x_i$$

or

$$\begin{aligned} &(m-1)\,d + \sum_{i=1}^{k} x_i b_i + b_1 + u'b_k \leqslant x \\ &\quad\leqslant (m-1)\,d + \sum_{i=1}^{k} x_i b_i + b_1 + u'b_k + h - \sum_{i=1}^{k} x_i - 1 - u' \end{aligned}$$

is contained in hA. It therefore follows from Lemma 1.1 that any x satisfying that

$$md+\sum_{i=1}^{k} x_i b_i \leqslant x \leqslant (m-1)\,d+\sum_{i=1}^{k} x_i b_i + b_1 + u' b_k - \sum_{i=1}^{k} x_i - 1 - u'$$

is contained in hA. But

$$(m-1)\,d+\sum_{i=1}^{k} x_i b_i + b_1 + u' b_k + h - \sum_{i=1}^{k} x_i - 1 - u' \geqslant md + \sum_{i=1}^{k} x_i b_i + b_1,$$

we therefore obtain that any x satisfying

$$md+\sum_{i=1}^{k} x_i b_i \leqslant x \leqslant md + (x_1+1)\,b_1 + \sum_{i=2}^{k} x_i b_i$$

is already contained in hA. The arbitrariness of x_1 implies that hA contains all integers in the interval

$$\left[md+\sum_{i=2}^{k} x_i b_i,\ md+ub_1+\sum_{i=2}^{k} x_i b_i\right].$$

Using Lemma 1.2, we have

$$md+\sum_{i=2}^{k} x_i b_i + ub_i + h - \sum_{i=2}^{k} x_i - u \geqslant md + (x_2+1)\,b_2 + \sum_{i=3}^{k} x_i b_i,$$

which implies that any x satisfying that

$$md+\sum_{i=2}^{k} x_i b_i \leqslant x \leqslant md + (x_2+1)\,b_2 + \sum_{i=3}^{k} x_i b_i$$

is contained in hA. Again by the arbitrariness of x_2, we have that hA contains all integers in the interval

$$\left[md+\sum_{i=3}^{k} x_i b_i,\ md+ub_2+\sum_{i=3}^{k} x_i b_i\right].$$

Again using Lemma 1.2, we have that hA contains all integers in the interval

$$\left[md+\sum_{i=3}^{k} x_i b_i,\ md+(x_3+1)\,b_3+\sum_{i=4}^{k} x_i b_i\right].$$

Using a similar argument, we obtain that any x satisfying

$$md \leqslant x \leqslant md + u' b_k$$

is contained in hA. Observing

$$\begin{aligned} b_1 - h + (k-1)\,u + u' &\leqslant \frac{2h}{k+1} - h + (k-1)\frac{h}{k+1} + \frac{2h}{3k+3} \\ &\leqslant \frac{2h}{3k+3} < h - u' - 1, \end{aligned}$$

we finally obtain that any x satisfying $md \leqslant x \leqslant (m+1)\,d$ is contained in hA for any $m > 1$. This means that $x \in hA$ for all $x > d$, i.e., $g(A) \leqslant h$, which ends the proof.

3. Estimate of $G_k(h)$ for Given h

In this section, $[a, b]$ denotes the set of all integers n such that $a \leqslant n \leqslant b$.

Lemma 3.1. *Let $h \geqslant 2$, $k \geqslant h-1$. Let $l = [k/(h-1)]$. Define*

$$b_0 = 1,$$

$$b_{il+j} = b_{il} + j\left(2 + \sum_{r=1}^{i} b_{rl}\right),$$

for $i = 0, 1, ..., h-2$ and $j = 1, 2, ..., l$. Let $m \geqslant 2$. Let

$$D = \{0, 1, md, md+1\}.$$

Then

$$\left[md + 2 + \sum_{r=1}^{i-1} b_{rl},\ md + 1 + \sum_{r=1}^{i} b_{rl}\right] \subseteq (i+1)(\{b_1, ..., b_{il}\} \cup D) \qquad (1)$$

holds for $i = 1, 2, ..., h-1$.

Proof. If $n \in [md+2,\ md+1+b_1]$, then

$$2 \leqslant n - md \leqslant b_1 + 1.$$

If $n - md$ is odd, then there exists a b_j $(1 \leqslant j \leqslant l)$ such that

$$b_j = n - md,$$

i.e., $a = md + b_j$. If $n - md$ is even, then $n = (md+1)+1$ or there exists a b_j such that $n = (md+1) + b_j$. Hence

$$n \in 2(\{b_1, ..., b_1\} \cup D).$$

Now assume (1) holds for any $i \leqslant s-1$, it follows that for any $i \leqslant s-1$,

$$\left[md, md+1+\sum_{r=1}^{i} b_{rl}\right] \subseteq (i+1)(\{b_1, ..., b_{il}\} \cup D).$$

Let

$$n \in \left[md+2+\sum_{r=1}^{s-2} b_{rl}, md+1+\sum_{r=1}^{s} b_{rl}\right].$$

If

$$md+2+\sum_{r=1}^{s-1} b_{rl} \leqslant n \leqslant md+b_{(s-1)l+1}-1,$$

then

$$md+2+\sum_{r=1}^{s-2} b_{rl} \leqslant n-b_{(s-1)l} \leqslant md+1+\sum_{r=1}^{s-1} b_{rl}.$$

It follows from the assumption that

$$n-b_{(s-1)l} \in (\{b_1, ..., b_{(s-1)l}\} \cup D),$$

which means $n \in (s+1)(\{b_1, ..., b_{(s-1)l}\} \cup D)$.

If there exists a j: $1 \leqslant j \leqslant l-1$ such that

$$md+b_{(s-1)l+j} \leqslant n \leqslant md+d_{(s-1)l+j+1}-1,$$

then

$$md \leqslant n-b_{(s-1)l+j} \leqslant md+1+\sum_{r=1}^{s-1} b_{rl}.$$

It follows from the assumption that

$$n \in (s+1)(\{b_1, ..., b_{(s-1)l}\} \cup D).$$

If

$$md+b_{sl} \leqslant n \leqslant md+1+\sum_{r=1}^{s} b_{rl},$$

then $n-b_{sl} \in s(\{b_1, ..., b_{(s-1)l}\} \cup D$, thus

$$n \in (s+1)(\{b_1, ..., b_{(s-1)l}\} \cup D).$$

The proof is complete.

The notation hA can be generalized. Let $A_1, ..., A_r$ be sets of integers. Define

$$\sum_{i=1}^{r} A_i = \left\{ \sum_{i=1}^{r} a_i \mid a_i \in A_i \text{ for } i = 1, 2, ..., r \right\}.$$

Let B be a set of integers, and let g be a positive integer. $B^{(g)}$ denotes the set of nonnegative integers congruent to some element of B. The notation $A \sim B$ means

$$A \cap \{n \geqslant N \mid n \in \mathbb{N}\} = B \cap \{n \geqslant N \mid n \in \mathbb{N}\}$$

for some N. $A(m)$ is the size of the set $\{a \in A \mid a \leqslant m\}$. The lower density of A is defined as

$$dA = \liminf_{m \to \infty} \frac{A(m)}{m}.$$

To give an upper bound for $G_k(h)$, we need the following Kneser's Theorem.

KNESER'S THEOREM (see [4]). *Let*

$$C = \sum_{i=1}^{n} A_i.$$

Then either

$$dC \geqslant \sum_{i=1}^{n} dA_i$$

or

$$C \sim C^{(g)}$$

for some g.

Now we give an upper bound for $G_k(h)$.

LEMMA 3.2. *Let* $h \geqslant 2$. *Then*

$$G_k(h) + 1 \leqslant 2k^{h-1}/(h-1) + (h+1)\, k^{(h-2)}/(h-2) + O(k^{h-3})$$

as k *tends to infinity.*

Proof. Let A be an asymptotic basis of order h. Let $F \in I_k(A)$. Let $B = A \setminus F$. We have

$$hB \cup F + (h-1)\, B \cup \cdots \cup (h-1)\, F + B \cup hF \sim N. \tag{2}$$

Hence $dhB > 0$ and

$$dhB + kd(h-1)\,B + \binom{k+1}{2} d(h-2)\,B + \cdots + \binom{k+h-2}{h-1} dB \geqslant 1.$$

Let $k \geqslant \max(h-1, 2)$. Then we have

$$\binom{k+h-1}{k} - 1 \geqslant h.$$

Therefore

$$d\left(\binom{k+h-1}{k} + 1\right) B + dhB + kd(h-1)\,B$$
$$+ \binom{k+1}{2} d(h-2)\,B + \cdots + \binom{k+h-2}{h-1} dB > 1.$$

It follows from Kneser's theorem that we have a g such that

$$\left(\binom{k+h-1}{k} - 1 + \sum_{i=0}^{h-1} \binom{k+i-1}{i} (h-i)\right) B$$
$$\sim \left(\binom{k+h-1}{k} - 1 + \sum_{i=0}^{h-1} \binom{k+i-1}{i} (h-i)\right) B^{(g)}.$$

Take the smallest g for which this last relation holds. We show that $g = 1$.
It follows from (2) that

$$hB^{(g)}(g) + k((h-1)\,B^{(g)}(g)) + \cdots + \binom{k+h-2}{h-1} (B^{(g)}(g)) \geqslant g.$$

Using Kneser's theorem, we have

$$\left(\sum_{i=0}^{h-1} \binom{k+i-1}{i} (h-i)\right) B^{(g)}(g) \geqslant g - \binom{h+k-1}{k} + 1.$$

It is clear that if $nB^{(g)}(g) < g$, then $(n+1)\,B^{(g)}(g) > nB^{(g)}(g)$. Therefore

$$\left(\binom{h+k-1}{k} - 1 + \sum_{i=0}^{h-1} \binom{k+i-1}{i} (h-i)\right) B^{(g)}(g) = g.$$

This implies $g = 1$. Therefore

$$G_k(h) + 1 \leqslant \binom{h+k-1}{k} + \sum_{i=0}^{h-1} \binom{k+i-1}{i} (h-i)$$
$$= \frac{2}{(h-1)!} k^{h-1} + \frac{h+1}{(h-2)!} k^{h-2} + O(k^{h-3})$$

as k tends to infinity. This completes the proof.

THEOREM 3.3. *Let $h \geqslant 2$. Then*

$$\begin{aligned} G_k(h)+1 &\geqslant 2(k/(h-1))^{h-1}+(4h-5)(k/(h-1))^{h-2}+O(k^{h-3}); \\ G_k(h)+1 &\leqslant 2k^{h-1}/(h-1)!+(h-1)\,k^{h-2}/(h-2)!+O(k^{h-3}) \end{aligned} \tag{3}$$

as k tends to infinity. In particular, $G_k(2)=2k+2$ holds for any $k \geqslant 1$.

Proof. The second inequality has already been established in Lemma 3.2. We now prove the first one.

Suppose $k \geqslant h-1$. Let $1, b_0, ..., b_{(h-1)l}$ be as in Lemma 3.1. Define

$$b_{(h-1)l+j}=b_{(h-1)l}+j\left(2+\sum_{r=1}^{h-2} b_{rl}\right)$$

for $j=1, 2, ..., k-(h-1)\,l$. Define

$$d=b_k+2+\sum_{r=1}^{h-2} b_{rl}.$$

Let

$$\begin{aligned} F&=\{b_1, b_2, ..., b_k\}, \\ A&=F \cup \{a \in \mathbb{N} \mid a \equiv 0 \text{ or } 1 \pmod d\}. \end{aligned}$$

It is clear that both A and $A \backslash F$ are asymptotic bases.

We prove that A is of order h. Let $m \geqslant 0$ be any integer. By Lemma 3.1 we have

$$\left[md+2+\sum_{r=1}^{s-1} b_{rl}, md+1+\sum_{r=1}^{s} b_{rl}\right] \subseteq (s+1)\,A$$

for $s=1, 2, ..., h-1$, which implies that

$$\left[md, md+1+\sum_{r=1}^{h-1} b_{rl}\right] \subseteq hA.$$

Let

$$n \in \left[md+2+\sum_{r=1}^{h-1} b_{rl}, md+d-1\right].$$

Suppose

$$md+b_{(h-1)l}+t\left(2+\sum_{r=1}^{h-2} b_{rl}\right) \leqslant n < md+b_{(h-1)l}+(t+1)\left(2+\sum_{r=1}^{h-2} b_{rl}\right)$$

for some t. Then

$$n-b_{(h-1)l+1}=n-\left((b_{(h-1)l}+t\left(2+\sum_{r=1}^{h-2} b_{rl}\right)\right)$$

is contained in $[md, md+1+\sum_{r=1}^{h-2} b_{rl}]$, hence in $(h-1)A$. This means $n \in hA$. Therefore $[md, (m+1)d-1] \subseteq hA$. Consequently, $g(A) \leqslant h$.

Since $A \setminus F=\{a \in N \mid a \equiv 0 \text{ or } 1 \pmod d\}$, it follows that $g(A \setminus F)=d-1$. Hence $G_k(h)+1 \geqslant d$. From the definition, a simple calculation shows that

$$b_{sl}=lb_{(s-1)l}+(b_{(s-1)l}+lb_{(s-2)l})+O(l^{s-2})$$

holds for $s=2, 3, ..., h-1$. This implies that

$$b_{(h-1)l}=2l^{h-1}+(4h-7)\,l^{h-2}+O(l^{h-3}).$$

Therefore

$$\begin{aligned}
G_k(h)+1 \geqslant d &= b_k+2+\sum_{r=1}^{h-2} b_{rl}\\
&= b_{(h-1)l}+(k-(h-1)\,l)\left(2+\sum_{r=1}^{h-2} b_{rl}\right)+2+\sum_{r=1}^{h-2} b_{rl}\\
&= 2l^{h-1}+(4h-7)\,l^{h-2}+b_{(h-2)l}+2+\sum_{r=1}^{h-3} b_{rl}\\
&\quad + (k-(h-1)\,l)\left(2+\sum_{r=1}^{h-2} b_{rl}\right)+O(l^{h-3})\\
&\geqslant 2l^{h-1}+(4h-5)\,l^{h-2}+2l^{h-2}(k-(h-1)\,l)+O(l^{h-3})\\
&\geqslant 2(k/(h-1))^{h-1}+(4h-5)(k/(h-1))^{h-2}\\
&\quad - 2(h-1)(k/(h-1)-1)(k/(h-1))^{h-2}\\
&\quad + 2(k-(h-1)\,l)(k/(h-1))^{h-2}+O(k^{h-3})\\
&= 2(k/(h-1))^{h-1}+(4h-5)(k/(h-1))^{h-2}+O(k^{h-3}),
\end{aligned}$$

which proves the inequality.

From Lemma 3.2 and the argument above, we see that the lower bound and the upper bound are polynomials of k with integral coefficients. Hence when $h=2$, the O's in (3) are zero, namely, $G_k(2)=2k+2$. It is readily seen that this holds for any $k \geqslant 2$, hence for any $k \geqslant 1$ since $G_1(2)=4$. (see [1]). The proof is complete.

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